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BOUND STATES IN GALILEAN-INVARIANT QUANTUM FIELD THEORY [1]

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Abstract

We consider the nonrelativistic quantum mechanics of a model of two spinless fermions interacting via a two-body potential. We introduce quantum fields associated with the two particles as well as the expansion of these fields in asymptotic "in" and "out" fields, including such fields for bound states, in principle. We limit our explicit discussion to a two-body bound state. In this context we discuss the implications of the Galilean invariance of the model and, in particular, show how to include bound states in a strictly Galilean-invariant quantum field theory.

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I. Introduction

The representations of the Galilean group relevant to quantum theory are ray representations with a non-trivial phase discovered by Bargmann[1]. The Bargmann phase leads to the Bargmann mass superselection rule that the mass of a bound state must be exactly the sum of the masses of its constituents. We analyze bound states in strictly Galilean-invariant theories taking account of the Bargmann phases. Our technique is the Haag expansion[2] of the fields that appear in the Hamiltonian in normal-ordered products of asymptotic (in- or out-) fields. We use the representation theory of the Galilean group due to Bargmann to constrain the form of the Haag expansion and derive the Schrödinger equation for bound states, the unitarity relation for elastic scattering and other relations in a unified way. We derive the relation between the in- and out- asymptotic fields for bound states by constructing the out-field as the asymptotic limit of the product of the fields of the constituents at separated points integrated with the bound state amplitude that serves as the wavefunction.

In Sec. II, we introduce the two-body model that we consider. In Sec. III, we derive the transformation properties of the Haag amplitudes under Galilean transformations. We take care to show that the Bargmann mass-dependent phases that occur in Galilean-invariant theories cancel so that we can consider breakup and rearrangement processes in which the initial particles have different masses from the final particles. As just mentioned, the Bargmann superselection rule requires that the sum of the masses that occur in the $\mathbf{p}^2/2m$ kinetic terms is absolutely conserved in all processes. In Sec. IV, we investigate the anticommutation relations of the interacting fields in terms of their Haag expansions and obtain relations (as examples, the relation between the bound-state amplitudes with different legs on- and off-shell, and elastic unitarity) among the amplitudes independent of the specific dynamics of a given theory. In Sec. V, we apply the NQA to the bound state problem and show that the Haag amplitude describing this state is just the Schrödinger wavefunction. To our knowledge this is the first description of a bound state in which Galilean invariance is strictly maintained. In Sec. VI, we apply the NQA to the two-body scattering problem and show that the corresponding amplitude yields the T-matrix after removal of its off-shell leg. In Sec. VII, we construct the asymptotic fields for a bound state as the integral with the bound-

state wavefunction of the product of the fields for the elementary constituents of the composite system. Here we differ from the proposals of Nishijima[3] and of Zimmermann[4] who construct bound states as products of the constituent fields at the same point. Section VIII concludes with a summary and the outlook for further research.

II. Two-Body Model

We consider a model in the Heisenberg picture with two spinless nonrelativistic Fermi fields, $A(\mathbf{x},t)$ and $B(\mathbf{x},t)$, with the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2m_A}\int d^3x\,\nabla_{\mathbf{x}}A^\dagger(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}A(\mathbf{x},t)+\frac{1}{2m_B}\int d^3x\,\nabla_{\mathbf{x}}B^\dagger(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}B(\mathbf{x},t) \\ &\quad +\int d^3x\,d^3y\,B^\dagger(\mathbf{y},t)A^\dagger(\mathbf{x},t)V_{AB}(|\mathbf{x}-\mathbf{y}|)A(\mathbf{x},t)B(\mathbf{y},t); \end{aligned} \tag{1}$$

for simplicity we assumed an AB interaction, but no AA or BB interaction. We assume V is smooth, not too long ranged, and not too singular at the origin. We want V to be well-behaved enough that the weak asymptotic limit we introduce just below exists. Since this is a very technical issue, we are deliberately vague about the necessary conditions. The literature on this issue can be traced from articles and references in[5]. The equation of motion for $A(\mathbf{x},t)$ is

$$i\partial_t A(\mathbf{x},t)=-\frac{1}{2m_A}\nabla^2_{\mathbf{x}}A(\mathbf{x},t)+\int d^3y\,B^\dagger(\mathbf{y},t)V_{AB}(|\mathbf{x}-\mathbf{y}|)B(\mathbf{y},t)A(\mathbf{x},t). \tag{2}$$

Some calculations are simpler in momentum space, therefore we define

$$A(\mathbf{x},t)=\int d^3k dE e^{-i(Et-\mathbf{k}\cdot\mathbf{x})}\tilde{A}(\mathbf{k},E), \tag{3}$$

$$V_{AB}(|\mathbf{x}-\mathbf{y}|)=\frac{1}{(2\pi)^3}\int d^3q\,e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}\tilde{V}_{AB}(|\mathbf{q}|). \tag{4}$$

Transforming the equation of motion to momentum space yields

$$(E_A-\frac{\mathbf{k}_A^2}{2m_A})\tilde{A}(\mathbf{k}_A,E_A)=\int dE_B d^3k_B dE'_B d^3k'_B dE'_A d^3k'_A$$

$$\times\delta(E_A+E_B-E'_B-E'_A)\delta(\mathbf{k}_A+\mathbf{k}_B-\mathbf{k}'_A-\mathbf{k}'_B)$$

$$\times \tilde{B}^{\dagger}(\mathbf{k}_B, E_B)\tilde{V}_{AB}(|\mathbf{k}'_B - \mathbf{k}_B|)\tilde{B}(\mathbf{k}'_B, E'_B)\tilde{A}(\mathbf{k}'_A, E'_A). \tag{5}$$

The asymptotic (in- or out-) fields for (possibly composite) particles are characterized by their rest energy E, mass m, and spin J. We will suppress the spin in what follows. The definitions of the asymptotic fields associated with the interacting field $A(\mathbf{x}, t)$ are

$$A^{in\ (out)}(\mathbf{x}, t) = lim_{t'\to\mp\infty} \int \mathcal{D}(\mathbf{x} - \mathbf{y}, t - t'; 0, m_A)A(\mathbf{y}, t')d^3y, \tag{6}$$

where the limit is the weak limit of the smeared operators, and

$$\mathcal{D}(\mathbf{x}, t; E, m) = \frac{1}{(2\pi)^3} \int d\omega d^3k\delta(\omega - E - \frac{\mathbf{k}^2}{2m})e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}. \tag{7}$$

The asymptotic fields for $B(\mathbf{x}, t)$ are defined in an analogous way. We give the definition of the asymptotic fields for composite particles in Sec. VII. The asymptotic limit in momentum space is often useful in calculations,

$$\tilde{A}^{in\ (out)}(\mathbf{k}, E) = lim_{t'\to\mp\infty}\delta(E - \mathbf{k}^2/2m) \int dE' e^{i(E-E')t'} \tilde{A}(\mathbf{k}, E'). \tag{8}$$

Either form of the definition of the asymptotic limits makes clear that the asymptotic fields obey the free equations of motion,

$$i\partial_t C^{in(out)}(\mathbf{x}, t) = (E - \frac{1}{2m}\nabla^2)C^{in(out)}(\mathbf{x}, t) \tag{9}$$

and also the free field anticommutation or commutation relations,

$$[C^{in(out)}(\mathbf{x}, t), C^{\dagger in(out)}(\mathbf{y}, t')]_{\pm} = \mathcal{D}(\mathbf{x} - \mathbf{y}, t - t'; E, m). \tag{10}$$

Note that

$$\mathcal{D}(\mathbf{x}, 0; E, m) = \delta(\mathbf{x}), \quad \forall E, m. \tag{11}$$

Using translation invariance, the Haag expansion of the interacting field $A(\mathbf{x}, t)$ in terms of in-fields takes the following form in position space (with an analogous expansion for the B field)

$$A(\mathbf{x}, t) = A^{in}(\mathbf{x}, t) + \sum_i \int d^3x_B d^3x_i f_{B;i}(\mathbf{x} - \mathbf{x}_B, t - t_B; \mathbf{x} - \mathbf{x}_i, t - t_i) \tag{12}$$

$$\times B^{\dagger\, in}(\mathbf{x}_B, t_B)(ABi)^{in}(\mathbf{x}_i, t_i)$$

$$+\int d^3x_B d^3x' d^3x'_B f_{B;AB}(\mathbf{x}-\mathbf{x}_B, t-t_B; \mathbf{x}-\mathbf{x}', t-t'_B; \mathbf{x}-\mathbf{x}'_B, t-t'_B)$$
$$\times B^{\dagger\, in}(\mathbf{x}_B, t_B) A^{in}(x', t') B^{in}(\mathbf{x}'_B) + \cdots. \tag{13}$$

Because the asymptotic fields obey free equations, the Haag amplitudes obey free equations in each individual argument. A simple way to see this is to note that the convolution in position space becomes a product in momentum space, so that the momentum arguments of the Haag amplitudes are multiplied by energy shell delta functions contained in the asymptotic fields. Thus only the on-energy shell part of the Haag amplitudes enters and the position space Haag amplitudes obey the free equations. Since both the asymptotic fields and the Haag amplitudes obey the free equations, the integrals are independent of the times t_B, t_i and of t_B, t', t'_B because of the translation invariance of the Schrödinger scalar products. We label the Haag amplitude that is the coefficient of a product of (asymptotic) creation and annihilation operators by the labels of the operators; the two-body (AB) bound state in level i is labeled by i.

We define

$$C^{in}(\mathbf{x}, t) = (2\pi)^{-3/2} \int dE d^3k \delta(E - E_C - \mathbf{k}^2/2m_C) e^{-iEt + i\mathbf{k}\cdot\mathbf{x}} c^{in}(\mathbf{k}), \tag{14}$$

$$[c^{in}(\mathbf{k}), c^{\dagger\ in}(\mathbf{l})]_+ = \delta(\mathbf{k}-\mathbf{l}) \tag{15}$$

$$f_{B;i}(\mathbf{x}, t; \mathbf{x}', t') =$$

$$\frac{1}{(2\pi)^3} \int d^3k_1 d^3k_2 \exp(i\frac{\mathbf{k}_1^2}{2m_B} t - i\mathbf{k}_1\cdot\mathbf{x} - i(-\epsilon_i + \frac{\mathbf{k}_2^2}{2m_{AB}})t' + i\mathbf{k}_2\cdot\mathbf{x}') \tilde{f}_{B;i}(\mathbf{k}_1, \mathbf{k}_2) \tag{16}$$

and similar definitions for other Fourier transforms chosen so that powers of 2π are absent from the momentum-space formulas. The result is

$$\tilde{A}(\mathbf{k}, E) = (2\pi)^{-3/2} a^{in}(\mathbf{k}) \delta(E - \frac{\mathbf{k}^2}{2m_A})$$

$$+\int d^3k_B d^3k_i\, \delta(E + \frac{\mathbf{k}_B^2}{2m_B} + \epsilon_i - \frac{\mathbf{k}_i^2}{2m_{AB}}) \delta(\mathbf{k} + \mathbf{k}_B - \mathbf{k}_i) \tilde{f}_{B;i}(\mathbf{k}_B; \mathbf{k}_i)$$

$$\times a^{in\dagger}(\mathbf{k}_B) a_i^{in}(\mathbf{k}_i)$$

$$+\int d^3k_B d^3k'_B d^3k'\,\delta(E+\frac{\mathbf{k}_B^2}{2m_B}-\frac{\mathbf{k}'^{\,2}}{2m_A}-\frac{\mathbf{k}_B'^{\,2}}{2m_B})\delta(\mathbf{k}+\mathbf{k}_B-\mathbf{k}'-\mathbf{k}'_B)$$

$$\times\,\tilde{f}_{B;AB}(\mathbf{k}_B;\mathbf{k}',\mathbf{k}'_B)a^{in\dagger}(\mathbf{k}_B)a^{in}(\mathbf{k}')a^{in}(\mathbf{k}_B)+\cdots. \tag{17}$$

(We use the abbreviation $m_{AB}=m_A+m_B$.) Note that we are expanding in terms of in-fields; there are analogous expansions in terms of out-fields. In the next section we derive the constraints on the f's that follow from Galilean invariance.

III. Galilean Invariance

Bargmann showed that the unitary projective representations (i.e., representations up to a factor) of the Galilean group that occur in the quantum mechanics of nonrelativistic particles cannot be reduced to vector (i.e., true) representations. This contrasts with the situation for the Poincaré and Lorentz groups, and–indeed–most other physically interesting groups, where the representations can be reduced to true representations. As already mentioned twice, the explicit mass parameter in the phases leads to the Bargmann superselection rule that the sum of the masses (that appear in the kinetic terms) must be conserved in every process. Nonetheless, bound states can be formed and particles can be created and annihilated, provided the Bargmann superselection rule is obeyed.

Note, for example, that if we were to assign rest energies m_A and m_B to particles A and B then a bound state of these particles with binding energy ϵ would have energy $E=m_{AB}-\epsilon+\mathbf{k}^2/2m_{AB}$, rather than $E=m_{AB}-\epsilon+\mathbf{k}^2/2(m_{AB}-\epsilon)$ as one might expect from the nonrelativistic limit of a relativistic bound state with rest energy $m_{AB}-\epsilon$. Another manifestation of this effect is that for this same bound state the momentum would transform under Galilean boosts as $\mathbf{k}\to\mathbf{k}+m_{AB}\mathbf{v}$, rather than as $\mathbf{k}\to\mathbf{k}+(m_{AB}-\epsilon)\mathbf{v}$.

If the projective representation has the form

$$U(G_2)U(G_1)=\omega(G_2,G_1)U(G_2G_1) \tag{18}$$

then another projective representation is equivalent to this if the other representation has the factor system $\omega'(G_2,G_1)=[\phi(G_2)\phi(G_1)/\phi(G_2G_1)]\omega(G_2,G_1)$, where ϕ has modulus one. This arbitrariness allows simplification of some formulas.

Bargmann gives as the Galilean transformation of a nonrelativistic scalar wave function,

$$(T(G)\psi)(\mathbf{x},t) = e^{-i\theta(G,(\mathbf{x},t))}\psi(G^{-1}(\mathbf{x},t)), \tag{19}$$

where $G(\mathbf{x},t) = (R\mathbf{x}+\mathbf{v}t+\mathbf{a}, t+b)$ and $\theta(G,(\mathbf{x},t)) = m(\frac{1}{2}\mathbf{v}^2 t - \mathbf{v}\cdot\mathbf{x})$. The Galilean transformation is labeled by $(\mathbf{a}, b, R, \mathbf{v})$, where $\mathbf{a}$ and b are space and time translations, R is a rotation and $\mathbf{v}$ is a boost. To infer the corresponding transformation for a nonrelativistic scalar field, we require

$$U(G)A(\psi)U^\dagger(G) = A(\psi_G), \quad \psi_G(\mathbf{x},t) = (T(G)\psi)(\mathbf{x},t) = e^{-i\theta(G,(\mathbf{x},t))}\psi(G^{-1}(\mathbf{x},t)), \tag{20}$$

$$A(\psi) = \int A(\mathbf{x},t)\psi(\mathbf{x},t)dt d^3x. \tag{21}$$

We find that

$$\begin{aligned} U(G)A(\mathbf{x},t)U^\dagger(G) &= e^{-i\theta_A(G,G(\mathbf{x},t))}A(G(\mathbf{x},t)), \\ \theta_A(G,G(\mathbf{x},t)) &= m_A[\frac{1}{2}\mathbf{v}^2(t+b) - \mathbf{v}\cdot(R\mathbf{x}+\mathbf{v}t+\mathbf{a})]. \end{aligned} \tag{22}$$

If the field has spin s, then A on the left hand side is replaced by A_i and A on the right hand side is replaced by $\sum_j A_j D^{(s)}_{ji}(G)$, where $D^{(s)}$ is a representation of $SU(2)$, which is the little group in this case. The corresponding transformation holds for B with m_B replacing m_A. Asymptotic fields transform the same way. The implications of the transformation law for the Haag amplitudes is found by transforming the interacting field in two ways: (1) act on the Haag expansion with $U(G)$ as in the left hand side of Eq.(22) and redefine the integration variables, and (2) multiply the Haag expansion by the phase factor on the right hand side of Eq.(22) and replace $(\mathbf{x},t)$ by $G(\mathbf{x},t)$. The two amplitudes $f_{B;i}$ and $f_{B;AB}$ obey

$$\begin{gathered} f_{B;i}(G(\mathbf{x}_A-\mathbf{x}_B, t_A-t_B); G(\mathbf{x}_A-\mathbf{x}_i, t_A-t_i)) = \\ e^{i\theta_A(G,G(\mathbf{x}_A,t_A))+i\theta_B(G,G(\mathbf{x}_B,t_B))-i\theta_{AB}(G,G(\mathbf{x}_i,t_i))} f_{B;i}(\mathbf{x}_A-\mathbf{x}_B, t_A-t_B); \mathbf{x}_A-\mathbf{x}_i, t_A-t_i), \end{gathered} \tag{23}$$

$$\begin{gathered} f_{B;AB}(G(\mathbf{x}_A-\mathbf{x}_B,, t_A-t_B)); G(\mathbf{x}_A-\mathbf{x}'_A, t_A-t'_A), G(\mathbf{x}_A-\mathbf{x}'_B, t_A-t'_B) = \\ e^{i\theta_A(G,G(\mathbf{x}_A,t_A))+i\theta_B(G,G(\mathbf{x}_B,t_B))-i\theta_A(G,G(\mathbf{x}'_A,t'_A))-i\theta_B(G,G(\mathbf{x}'_B),t'_B))} \\ \times f_{B;AB}(\mathbf{x}_A-\mathbf{x}_B, t_A-t_B; x_A-\mathbf{x}'_A, t_A-t'_A), \mathbf{x}_A-\mathbf{x}'_B, t_A-t'_B). \end{gathered} \tag{24}$$

Note that θ_{AB} is independent of the bound state i because of the Bargmann mass superselection rule. The combination of phases in the first of these is

$$\theta_A(G, G(\mathbf{x}_A, t_A)) + \theta_B(G, G(\mathbf{x}_B, t_B)) - \theta_{AB}(G, G(\mathbf{x}_i, t_i)) =$$

$$-\frac{1}{2}\mathbf{v}^2(m_A t_A + m_B t_B - m_{AB} t_i) - \mathbf{v} \cdot R(m_A \mathbf{x}_A + m_B \mathbf{x}_B - m_{AB} \mathbf{x}_i). \qquad (25)$$

The transformation law is *not* satisfied by having a delta function in the space and time coordinates identifying the coordinates $(\mathbf{x}_i, t_i))$ with the center-of-mass of particles A and B, although at equal times such a delta function does occur for the space coordinates. The way in which the transformation laws are satisfied is best seen in momentum space, where the corresponding transformations in momentum space are

$$(V(G)\phi)(\mathbf{k}, E) = e^{-i\Omega(G,(\mathbf{k},E))}\phi(G^{-1}(\mathbf{k}, E)), \qquad (26)$$

$$\Omega(G, (\mathbf{k}, E)) = (\mathbf{k} - m\mathbf{v}) \cdot \mathbf{a} - (E - \frac{1}{2}m\mathbf{v}^2)b, \qquad (27)$$

where $G(\mathbf{k}, E) = (R\mathbf{k}+m\mathbf{v}, E+\mathbf{v}\cdot R\mathbf{k}+\frac{1}{2}m\mathbf{v}^2)$, and $G^{-1}(\mathbf{k}, E) = (R^{-1}(\mathbf{k}-m\mathbf{v}), E-\mathbf{k} \cdot \mathbf{v} + \frac{1}{2}m\mathbf{v}^2)$. The momentum space transformation law for the field is induced in parallel with the derivation of the position space law. The result is

$$W(G)A(\mathbf{k}, E)W^{\dagger}(G) = e^{-i\Omega_A(G,-G(\mathbf{k},E))}A(G(\mathbf{k}, E), \qquad (28)$$

where $\Omega_A(G, -G(\mathbf{k}, E)) = (E + \mathbf{v} \cdot R\mathbf{k})b - R\mathbf{k} \cdot \mathbf{a}$. In the transformation law for the Haag amplitudes, all the phase factors cancel and the result for–say–the second term in the Haag expansion is what one would expect naively,

$$\tilde{f}_{B;i}(\mathbf{k}_B; \mathbf{k}_i) = \tilde{f}_{B;i}(R(\mathbf{k}_B - m_B\mathbf{v}); R(\mathbf{k}_i - m_{AB}\mathbf{v})). \qquad (29)$$

Thus we can choose the $\mathbf{v} = \mathbf{k}_i/m_{AB}$ so that the bound-state momentum vanishes and eliminate the second argument of $f_{B;i}$,

$$\tilde{f}_{B;i}(\mathbf{k}_B; \mathbf{k}_i) = \tilde{f}_{B;i}(\mathbf{k}_B - \frac{m_B}{m_{AB}}\mathbf{k}_i, \mathbf{0}) \equiv \tilde{F}_{B;i}(\mathbf{k}_B - \frac{m_B}{m_{AB}}\mathbf{k}_i). \qquad (30)$$

For the spinless case, $\tilde{F}_{B;i}(\mathbf{k}) = \tilde{F}_{B;i}(R\mathbf{k})$. All these results are exact, valid in any Galilean frame. The extension to fields with spin is straightforward. It is worth noting that the Poincaré transformation law in a relativistic theory is simpler than

the Galilean transformation law we have just derived for a nonrelativistic theory, because the Bargmann phase is absent for the Poincaré group.

Taking account of Galilean invariance, the position-space Haag amplitude is

$$f_{B;i}(\mathbf{x},t;\mathbf{x}',t') = (2\pi)^{-3}\int d^3k d^3k_i exp[i(m_B+\frac{1}{2m_B}(\mathbf{k}+\frac{m_B}{m_{AB}}\mathbf{k}_i)^2)t-i(\mathbf{k}+\frac{m_B}{m_{AB}}\mathbf{k}_i)\cdot\mathbf{x}]$$

$$\times\, exp[-i(-\epsilon_i+\frac{\mathbf{k}_i^2}{2m_{AB}})t'+i\mathbf{k}_i\cdot\mathbf{x}']\times\tilde{f}_{B;i}(\mathbf{k};\mathbf{0}). \tag{31}$$

The integral over $\mathbf{k}_i$ can be done, but the result is complicated and not useful, except when all times are equal, in which case the result is both simple and useful,

$$f_{B;i}(\mathbf{x}_A-\mathbf{x}_B;\mathbf{x}_A-\mathbf{x}_i) = \delta(\mathbf{x}_i-\frac{m_A\mathbf{x}_A+m_B\mathbf{x}_B}{m_{AB}})F_{B;i}(\mathbf{x}_A-\mathbf{x}_B), \tag{32}$$

$$F_{B;i}(\mathbf{x}) = \int d^3k e^{-i\mathbf{k}\cdot\mathbf{x}}\tilde{f}_{B;i}(\mathbf{k};\mathbf{0}). \tag{33}$$

Using the constraints due to Galilean invariance, the Haag expansion in x-space at equal times takes the form

$$A(\mathbf{x}) = A^{in}(\mathbf{x})+\sum_i\int F_{B;i}(\mathbf{x}-\mathbf{x}_B)B^{\dagger in}(\mathbf{x}_B)(ABi)^{in}(\frac{m_A\mathbf{x}+m_B\mathbf{x}_B}{m_{AB}})d^3x_B$$

$$+\int d^3r'd^3rF_{B;AB}(\mathbf{r}';\mathbf{r})B^{\dagger in}(\mathbf{x}-\mathbf{r}')A^{in}(\mathbf{x}+\frac{m_B(\mathbf{r}-\mathbf{r}')}{m_{AB}})B^{in}(\mathbf{x}-\frac{m_A\mathbf{r}+m_B\mathbf{r}'}{m_{AB}})$$

$$+\cdots. \tag{34}$$

In momentum space, the expansion is

$$\tilde{A}(\mathbf{k},E) = \frac{1}{(2\pi)^{3/2}}a^{in}(\mathbf{k})\delta(E-\frac{\mathbf{k}^2}{2m_A})$$

$$+\int d^3k_B\delta(E+\frac{\mathbf{k}_B^2}{2m_B}+\epsilon_i-\frac{(\mathbf{k}+\mathbf{k}_B)^2}{2m_{AB}})\tilde{F}_{B;i}(\frac{m_A\mathbf{k}_B-m_B\mathbf{k}}{m_{AB}})b^{in\dagger}(\mathbf{k}_B)c_i^{in}(\mathbf{k}+\mathbf{k}_B)_i$$

$$+\int d^3k_Bd^3p_Bd^3p\delta(E+\frac{\mathbf{k}_B^2}{2m_B}-\frac{\mathbf{p}^2}{2m_A}-\frac{\mathbf{p}_B^2}{2m_B})\delta(\mathbf{k}+\mathbf{k}_B-\mathbf{p}-\mathbf{p}_B)$$

$$\times \tilde{F}_{B;AB}(\frac{m_A\mathbf{k}_B - m_B\mathbf{k}}{m_{AB}}; \frac{m_A\mathbf{p}_B - m_B\mathbf{p}}{m_{AB}})b^{in\ \dagger}(\mathbf{k}_B)a^{in\ \dagger}(p)b^{in}(\mathbf{p}_B) + \cdots. \quad (35)$$

Here c_i^{in} is the annihilation operator for the bound state of A and B in state i.

IV. Two-Body Bound State

To derive the equation for the two-body bound state, we insert the Haag expansion Eq.(13) in the equation of motion Eq.(2), renormal order, and equate the coefficients of the terms with the operators $B^{\dagger in}(AB_i)^{in}$. After commuting or anticommuting with the relevant in-fields, the result is

$$(i\frac{\partial}{\partial t} + \frac{1}{2m_A}\nabla_x^2 - V(|\mathbf{x} - \mathbf{x}_B|))f_{B;i}(\mathbf{x} - \mathbf{x}_B, t - t_B; \mathbf{x} - \mathbf{x}_i, t - t_i) = 0. \quad (36)$$

It is convenient to eliminate the time derivative by using $\partial/\partial t = -\partial/\partial t_B - \partial/\partial t_i$, the independence of the Schrödinger scalar product on the time and the free equations satisfied by the in-fields to find free equations for the t_B and t_i dependences of $f_{B;i}$. The results are

$$(i\frac{\partial}{\partial t_B} + \frac{1}{2m_B}\nabla_{x_B}^2)f_{B;i} = 0, \quad (37)$$

$$(i\frac{\partial}{\partial t_i} - \epsilon_i - \frac{1}{2m_{AB}}\nabla_{x_i}^2)f_{B;i} = 0. \quad (38)$$

The equation without time derivatives is

$$[-\frac{1}{2m_A}\nabla_x^2 - \frac{1}{2m_B}\nabla_{x_B}^2 + V(|\mathbf{x} - \mathbf{x}_B|)]f_{B;i} = (\epsilon_i - \frac{1}{2m_{AB}}\nabla_{x_i}^2)f_{B;i}. \quad (39)$$

Now using Eq.(32) we find the usual Schrödinger equation for $F_{B;i}$,

$$[-\frac{1}{2\mu}\nabla_{r_{AB}}^2 + V(\mathbf{r}_{AB})]F_{B;i} = -\epsilon_i F_{B;i}, \quad \frac{1}{\mu} = \frac{1}{m_A} + \frac{1}{m_B}, \quad (40)$$

where the reduced mass enters. This establishes that $F_{B;i}$ is the Schrödinger wave function of the bound state. Note that the bound-state amplitude is given *exactly* in any reference frame in terms of the amplitude in the rest frame of the bound state. (The corresponding statement also holds for other amplitudes, as well as for relativistic theories.)

V. Two-Body Scattering

Two-body scattering is described in position space at equal times by the amplitude

$$f_{B;AB}(\mathbf{x}_A-\mathbf{x}_B,0;\mathbf{x}_A-\mathbf{y}_A,0,\mathbf{x}_B-\mathbf{y}_B,0)=F_{B;AB}(\mathbf{x}_A-\mathbf{x}_B;\mathbf{y}_A-\mathbf{y}_B)\delta(\mathbf{R}'-\mathbf{R}), \quad (41)$$

$$F_{B;AB}(\mathbf{x};\mathbf{y})=(2\pi)^{-3/2}\int d^3k'd^3k\tilde{f}_{B;AB}(\mathbf{k}';-\mathbf{k},\mathbf{k})exp\, i[-\mathbf{k}'\cdot(\mathbf{x}_A-\mathbf{x}_B)+\mathbf{k}\cdot(\mathbf{y}_A-\mathbf{y}_B)],$$
$$\mathbf{R}'=\frac{m_A\mathbf{x}_A+m_B\mathbf{x}_B}{m_{AB}}, \qquad \mathbf{R}=\frac{m_A\mathbf{y}_A+m_B\mathbf{y}_B}{m_{AB}}. \quad (42)$$

We prefer to discuss two-body scattering in momentum space, using the amplitude $\tilde{f}_{B;AB}(\mathbf{k}_B;\mathbf{p}_A,\mathbf{p}_B)$ which is the coefficient of the term $b_B^{in\dagger}(\mathbf{k}_B)a^{in}(\mathbf{p}_A)b^{in}(\mathbf{p}_B)$ in the Haag expansion of $A(\mathbf{k},E)$. The procedure for finding the equation for $\tilde{f}_{B;AB}$ is analogous to that for the two-body bound state amplitude. We find

$$\left(\frac{\mathbf{p}_A^2-(\mathbf{p}_A+\mathbf{p}_B-\mathbf{k}_B)^2}{2m_A}+\frac{\mathbf{p}_B^2-\mathbf{k}_B^2}{2m_B}\right)\tilde{f}_{B;AB}(\mathbf{k}_B;\mathbf{p}_A,\mathbf{p}_B) =$$
$$\tilde{V}_{AB}(|\mathbf{k}_B-\mathbf{p}_B|)+\int d^3k'_B\,\tilde{V}_{AB}(|\mathbf{k}_B-\mathbf{k}'_B|)\tilde{f}_{B;AB}(\mathbf{k}'_B;\mathbf{p}_A,\mathbf{p}_B). \quad (43)$$

Galilean invariance relates $\tilde{f}_{B;AB}$ at arbitrary momenta to itself in the center-of-mass,

$$\tilde{f}_{B;AB}(\mathbf{k}_B;\mathbf{p}_A,\mathbf{p}_B)=\tilde{f}_{B;AB}(R(\mathbf{k}_B-m_B\mathbf{v});R(\mathbf{p}_A-m_A\mathbf{v}),R(\mathbf{p}_B-m_B\mathbf{v})). \quad (44)$$

By choosing $\mathbf{v}=(\mathbf{p}_A+\mathbf{p}_B)/m_{AB}$, we can replace $\tilde{f}_{B;AB}$ by a function of one fewer variable,

$$\tilde{f}_{B;AB}(\mathbf{k}_B;\mathbf{p}_A,\mathbf{p}_B)=\tilde{F}_{B;AB}(\mathbf{k};\mathbf{p}), \quad (45)$$

where here and below, $\mathbf{k}=(m_A\mathbf{k}_B-m_B\mathbf{k}_A)/m_{AB}$, $\mathbf{p}=(m_A\mathbf{p}_B-m_B\mathbf{p}_A)/m_{AB}$ and we used conservation of momentum to introduce $\mathbf{k}_A$. The momenta $\mathbf{p}$ and $\mathbf{k}$ are the center-of-mass momenta of particle B in the initial and the final state, respectively. The elastic scattering equation becomes

$$\frac{1}{2\mu}(\mathbf{p}^2-\mathbf{k}^2)\tilde{F}_{B;AB}(\mathbf{k};\mathbf{p})=\tilde{V}(|\mathbf{k}-\mathbf{p}|)+\int d^3k'\tilde{V}(\mathbf{k}-\mathbf{k}')\tilde{F}_{B;AB}(\mathbf{k}';\mathbf{p}), \quad (46)$$

The solution is the Born series,

$$\tilde{F}_{B;AB}(\mathbf{k};\mathbf{p})=\tilde{G}_R(\mathbf{k};\mathbf{p})\tilde{V}(|\mathbf{k}-\mathbf{p}|)+ \quad (47)$$

$$\tilde{G}_R(\mathbf{k};\mathbf{p})\int d^3k'\tilde{V}(|\mathbf{k}-\mathbf{k}'|)\tilde{G}_R(\mathbf{k}';\mathbf{p})\tilde{V}(|\mathbf{k}'-\mathbf{p}|)+\cdots, \tag{48}$$

$\tilde{G}_R(\mathbf{k};\mathbf{p}) = [(\mathbf{p}^2-\mathbf{k}^2)/2\mu - i\epsilon]^{-1}$.

The amplitude $\tilde{F}_{B;AB}$ is closely related to the T-matrix element for AB scattering. The S-matrix element is

$$S(\mathbf{k}_A,\mathbf{k}_B;\mathbf{p}_A,\mathbf{p}_B) = \langle 0|b^{out}(\mathbf{k}_B)a^{out}(\mathbf{k}_A)a^{in\ \dagger}(\mathbf{p}_A)b^{in\ \dagger}(\mathbf{p}_B)|0\rangle. \tag{49}$$

In order to eliminate the out-fields in terms of the in-fields we need the definitions, given above in Eq.(6),

$$A^{in(out)}(\mathbf{x},t) = \lim_{\tau\to\mp\infty}\int_{t'=\tau} d^3x'\,\mathcal{D}(\mathbf{x}-\mathbf{x}',t-t';m_A,m_A)A(\mathbf{x}',t'), \tag{50}$$

where $\mathcal{D}$ was defined in Eq.(7). The nonrelativistic analog of the reduction formula follows from calculating $\int d^3x'dt'\partial/\partial_{t'}\mathcal{D}(\mathbf{x}-\mathbf{x}',t-t';m_A,m_A)A(\mathbf{x}',t')$ in two ways: performing the integral and carrying out the derivative. The result [5] is

$$A^{out}(\mathbf{x},t) - A^{in}(\mathbf{x},t) = \int d^3x'dt'\,\mathcal{D}(\mathbf{x}-\mathbf{x}',t-t';m_A,m_A)(\partial_{t'}-\frac{i}{2m_A}\nabla^2_{\mathbf{x}'})A(\mathbf{x}',t'). \tag{51}$$

Fourier transforming this yields

$$\frac{1}{(2\pi)^{3/2}}(a^{out}(\mathbf{k})-a^{in}(\mathbf{k})) = -2\pi i(E-\frac{\mathbf{k}^2}{2m_A})A(\mathbf{k},E). \tag{52}$$

Note that a factor of $\delta(E-\mathbf{k}^2/2m_A)$ has been removed from this equation; thus the right-hand-side is non-vanishing (and there is scattering) only when $A(\mathbf{k},E)$ has a pole at $E-\mathbf{k}^2/2m_A = 0$. Since $a^{\dagger\ out}(\mathbf{k})|0\rangle = a^{\dagger\ in}(\mathbf{k})|0\rangle$ for stable particles, the only out operator in the S-matrix element $\langle 0|b^{out}(\mathbf{k}_B)a^{out}(\mathbf{k}_A)A^{\dagger in}(\mathbf{p}_A)b^{\dagger in}(\mathbf{p}_B)|0\rangle$ that must be eliminated using Eq.(52) is $a^{out}(\mathbf{k}_A)$. The result is

$$S(\mathbf{k}_A,\mathbf{k}_B;\mathbf{p}_A,\mathbf{p}_B) = \delta(\mathbf{k}_A-\mathbf{p}_A)\delta(\mathbf{k}_B-\mathbf{p}_B) - 2\pi i\delta(\frac{\mathbf{k}_A^2}{2m_A}+\frac{\mathbf{k}_B^2}{2m_B}-\frac{\mathbf{p}_A^2}{2m_A}-\frac{\mathbf{p}_B^2}{2m_B})$$

$$\times\,\delta(\mathbf{k}_A+\mathbf{k}_B-\mathbf{p}_A-\mathbf{p}_B)(\frac{\mathbf{k}_A^2}{2m_A}+\frac{\mathbf{k}_B^2}{2m_B}-\frac{\mathbf{p}_A^2}{2m_A}-\frac{\mathbf{p}_B^2}{2m_B})\tilde{F}_{B;AB}(\mathbf{k};\mathbf{p}), \tag{53}$$

where again $\mathbf{k}$ and $\mathbf{p}$ are defined below Eq.(45). Thus the reduced T-matrix for elastic scattering on the momentum shell[6] is

$$t(\mathbf{k}_A, \mathbf{k}_B; \mathbf{p}_A, \mathbf{p}_B) = (\frac{\mathbf{p}_A^2}{2m_A} + \frac{\mathbf{p}_B^2}{2m_B} - \frac{\mathbf{k}_A^2}{2m_A} - \frac{\mathbf{k}_B^2}{2m_B})\tilde{F}_{B;AB}(\mathbf{k}; \mathbf{p}). \tag{54}$$

We emphasize that because the Haag amplitude is the scattering amplitude with one leg off shell, it contains the information necessary for calculations in the three-body sector. This contrasts with the on-shell scattering amplitude, which does not suffice for such calculations.

VI. Anticommutation Relations

In this section we show that the canonical (equal time) anticommutation relations of the Lagrangian fields imply general relations among Haag amplitudes, independent of the equations of motion of the specific theory. For example, the vanishing of the canonical anticommutator $[A, B]_+$ at equal times, considered for the coefficient of the bound state in-field for state i, gives

$$F_{A;i}(\mathbf{y} - \mathbf{x}) = F_{B;i}(\mathbf{x} - \mathbf{y}) \equiv F_i(\mathbf{x} - \mathbf{y}) \tag{55}$$

where we took $(ABi)^{in}(\mathbf{R}) = -(BAi)^{in}(\mathbf{R})$ because of the Fermi statistics of A and B. This shows that the apparent asymmetry in the treatment of the constituents of the bound state, due to the fact that the Haag amplitude that serves as the two-body wave function of the (AB) bound state in the Haag expansion of the A field has the A particle off-shell and the B particle on-shell, while these roles are interchanged for the amplitude for the same bound state in the Haag expansion of the B field, is not a real asymmetry. These two amplitudes determine each other uniquely. The analogous result for the off-shell elastic scattering amplitudes is

$$F_{B;AB}(\mathbf{x} - \mathbf{y}; \mathbf{r}) = F_{A;BA}(\mathbf{y} - \mathbf{x}; -\mathbf{r}) \equiv F_{AB}(\mathbf{x} - \mathbf{y}; \mathbf{r}). \tag{56}$$

Again the two apparently different off-shell amplitudes uniquely determine each other.

The consequence for elastic scattering is

$$t(\mathbf{k}_A, \mathbf{k}_B; \mathbf{p}_A, \mathbf{p}_B) - (t(\mathbf{p}_A, \mathbf{p}_B; \mathbf{k}_A, \mathbf{k}_B))^* =$$

$$(2\pi)^{5/2}\int d^3q_A d^3q_B \delta(\frac{\mathbf{k}_A^2}{2m_A}+\frac{\mathbf{k}_B^2}{2m_B}-\frac{\mathbf{q}_A^2}{2m_A}-\frac{\mathbf{q}_B^2}{2m_B})\delta(\mathbf{k}_A+\mathbf{k}_B-\mathbf{q}_A-\mathbf{q}_B)$$

$$\times t(\mathbf{k}_A,\mathbf{k}_B;\mathbf{q}_A,\mathbf{q}_B)(t(\mathbf{p}_A,\mathbf{p}_B;\mathbf{q}_A,\mathbf{q}_B))^*, \tag{57}$$

where $\mathbf{k}$ and $\mathbf{p}$ are as defined below Eq.(45). This is elastic unitarity.

The canonical anticommutator $[A, A^\dagger]_+$ at equal times leads to a generalization of unitarity,

$$\frac{1}{(2\pi)^{3/2}}(\tilde{F}_{B;AB}(\mathbf{k};\mathbf{p})+\tilde{F}^*_{B;AB}(\mathbf{p};\mathbf{k}))$$

$$=\sum_i \tilde{F}_{B;i}(\mathbf{k})\tilde{F}^*_{B;i}(\mathbf{p})+\int d^3q \tilde{F}_{B;AB}(\mathbf{k};\mathbf{q})\tilde{F}^*_{B;AB}(\mathbf{p};\mathbf{q}), \tag{58}$$

where again $\mathbf{k}$ and $\mathbf{p}$ are as defined below Eq.(45) and we have used momentum conservation, $\mathbf{k}_A+\mathbf{k}_B=\mathbf{p}_A+\mathbf{p}_B$. By taking the appropriate limit, we recover the elastic unitarity relation, Eq.(57). Taking into account the relations between the Haag amplitudes in the expansions of A and of B, these are all the independent two-body relations obtained from the anticommutation relations.

There are also quadratic relations between the amplitudes for the (ABi) and (ABj) bound states and the amplitudes for the breakup of these bound states due to scattering with the A or B particle. Since this involves a higher sector, we do not give this relation here.

VII. Construction of the asymptotic field for the bound state

In this section we show how to construct the asymptotic field for the bound state from a product of Lagrangian fields. The procedure is to multiply the appropriate Lagrangian fields at separated space points, integrate with the bound-state amplitude in the relative coordinate, and take the asymptotic limit. The result is

$$(ABi)^{in~(out)}(\mathbf{x},t)=\lim_{\tau\to\mp\infty}\int_{t'=\tau} d^3x'\,\mathcal{D}(\mathbf{x}-\mathbf{x}',t-t';-\epsilon_i,m_{AB})F_i^*(\mathbf{w})$$

$$\times\frac{1}{2}[B(\mathbf{y}-\frac{m_A}{m_{AB}}\mathbf{w},t'),A(\mathbf{y}+\frac{m_B}{m_{AB}}\mathbf{w},t')]_-d^3w. \tag{59}$$

A straightforward calculation shows these limits are $(ABi)^{in~(out)}(\mathbf{x},t)$ for $\tau\to\mp\infty$ and the leading terms for $\tau\to\pm\infty$ are $(ABi)^{out~(in)}(\mathbf{x},t)$. This is what we expect.

The higher terms in the Haag expansion for $(ABi)^{out\ (in)}((\mathbf{x},t)$ are in a higher sector that we don't discuss here.

VIII. Summary and outlook for further work

We have derived many results of the nonrelativistic quantum mechanics of two-particle systems in a unified way with particular attention to Galilean invariance, taking into account the fact that the representations of the Galilean group in quantum mechanics are necessarily representations up to a factor, rather than vector representations. We established the physical interpretation of the Haag amplitudes: the Haag amplitude for the simplest term with the two-body bound-state operator is precisely the Schrödinger wave function of the two-body bound state. This interpretation will carry over to explicitly covariant relativistic theories, where the corresponding Haag amplitude will be a three-dimensional object, but will be covariant. Of course in the relativistic case, a bound state that is mainly a two-body state will also have amplitudes to be composed of more particles. We constructed the asymptotic field for a composite particle as the weak limit of a product of the fields of the constituent particles weighted with the bound-state amplitude of the composite particle. We plan later to apply the N quantum formalism described here to several-particle systems, including scattering processes involving bound states and rearrangement collisions. In these cases, this formalism differs markedly from the usual methods, such as the Faddeev analysis of three-body problems. We do not discuss here the problems that arise when the number of particles increases without bound; for example, in the thermodynamic limit. See Narnhofer and Thirring[9] for a discussion. The use of our techniques in relativistic theories has been considered in[7] among other places. This reference shows that, at least in the weak-coupling approximation, the technique we discuss here can be used to find bound states in relativistic theories. We have also solved the Nambu–Jona-Lasinio model in one-loop approximation with the Haag expansion[8]. Although calculations based on the operator field equations are not the most popular way to study relativistic theories, the references just cited show that this can be a useful way to study such theories. We are presently studying approximations in which we don't assume weak coupling in collaboration with M. Malheiro and Y. Umino. We hope that this method will serve as an alternative to the Bethe-Salpeter equation in relativistic problems. We also plan to construct variational principles based on the Haag expansion for both

nonrelativistic and relativistic theories.

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